OPERATIONAL CALCULUS IN BANACH ALGEBRAS FOR ALGEBRA-VALUED FUNCTIONS[1]

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Introduction. We study an operational calculus in Banach algebras for *algebra-valued* locally holomorphic functions of a complex variable. This calculus is motivated by the operational calculus in Banach algebras for analytic functions of several complex variables (see [1; 2]).

Let A be a Banach algebra with unit e. Let $f(\lambda)$ be an A-valued function of the complex variable λ, locally holomorphic in an open set $\Delta(f)$. Then, for each x in A with spectrum in $\Delta(f)$, we define

$$f(x) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda;x)d\lambda$$

where $R(\lambda; x)$ is the resolvent of x and Γ is a suitable finite union of simple Jordan contours.

The natural questions about the above operational calculus are studied: multiplicativity of the map $f(\lambda)\rightarrow f(x)$; spectral mapping; composite A-valued functions. For example, we show that if the map $f(\lambda)\rightarrow f(x)$ is multiplicative over the functions $f(\lambda)$ with $\Delta(f)$ including the spectrum of x, then x is in the center of A. Concerning the spectral mapping problem, we may write the standard Spectral Mapping Theorem in the form

$$\sigma(f(x)) = \bigcup_{\lambda\in\sigma(x)} \sigma(f(\lambda)) \tag{1}$$

where $f(\lambda) = \phi(\lambda)e$ and $\phi(\lambda)$ is scalar-valued. We show that the inclusion $\subset$ is valid in the general case of A-valued functions if x commutes with $f(\lambda)$ for each λ. Furthermore, for μ not in the set on the right of (1), we have

$$R(\mu;f(x)) = \frac{1}{2\pi i}\int_{\Gamma_\mu} R(\mu;f(\lambda))R(\lambda; x)d\lambda \tag{2}$$

where Γ_μ is suitably chosen (Theorem 2).

This superposition formula has many applications which are discussed in §4. In particular, we obtain certain convolutive properties of the resolvent which

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are useful in perturbation problems and in the explicit representation of solutions of functional equations. Using (2), we prove also a "composite function" theorem (Theorem 4).

1. **Preliminaries.** Let A be a Banach algebra with unit e. For $x \in A$, $\sigma(x)$ denotes([2]) the spectrum of x, $\rho(x)$ the resolvent set of x, and $r(x)$ the spectral radius of x. For $\lambda \in \rho(x)$, $R(\lambda; x)$ denotes the resolvent of x, i.e., $R(\lambda; x) = (\lambda e - x)^{-1}$. If x and y are in A, the symbol $x \cup y$ means that x commutes with y.

We use freely integration of A-valued functions, referring to Hille and Phillips [4] for definitions and properties.

An A-valued function $f(\lambda)$ is *locally holomorphic* in the open subset Δ of the complex plane if each point of Δ has a connected neighborhood in Δ in which $f(\lambda)$ is holomorphic (i.e., single-valued and differentiable in the norm topology of A). The union of all the open sets in which $f(\lambda)$ is locally holomorphic is denoted by $\Delta(f)$.

The triplet (Φ, Δ, f) means the following:

(1) Φ is a compact subset of the complex plane;

(2) Δ is an open subset of the complex plane;

(3) $\Phi \subset \Delta \subset \Delta(f)$.

With Φ and Δ as above, we may associate an open subset Ω of the complex plane with the following properties:

(a) $\Phi \subset \Omega \subset \Delta$;

(b) Ω has a finite number of components Ω_i;

(c) each Ω_i is bounded by a finite number of simple closed rectifiable curves Γ_{ik};

(d) Ω has a positive distance from the boundary of Δ (see [4, p. 166]).

The boundary $\Gamma = \bigcup \Gamma_{ik}$ of Ω, with the orientation induced by the usual orientations of each Γ_{ik}, will be denoted by $\Gamma(\Phi, \Delta)$. This symbol represents any Γ with the above properties.

Definition 1. Consider the triplet $(\sigma(x), \Delta, f)$, where $x \in A$. The element $f(x)$ of A is defined by

$$f(x) = \frac{1}{2\pi i} \int_{\Gamma(\sigma(x), \Delta)} f(\lambda) R(\lambda; x) d\lambda.$$

The mapping $f(\lambda) \to f(x)$ has the following properties:

(P_0) The element $f(x) \in A$ exists and is well-defined; it depends only on $f(\lambda)$ and x, and not on the special choice of Γ, provided that $\Gamma = \Gamma(\sigma(x), \Delta)$.

(P_1) If $f(\lambda) \equiv y$, then $f(x) = y$.

(P_2) If $f(\mu) \equiv \lambda e$, then $f(x) = x$.

(P_3) Let $(\sigma(x), \Delta_i, f_i)$ $(i = 1,2)$ be triplets. Let $\Delta = \Delta_1 \cap \Delta_2$.

([2]) See [4] for definitions.

(A) For the triplet $(\sigma(x),\Delta,f)$ with $f(\lambda)=f_1(\lambda)+f_2(\lambda)$, we have $f(x)=f_1(x)+f_2(x)$.

(B) For the triplet $(\sigma(x),\ \Delta,\ h)$ with $h(\lambda)=f_1(\lambda)f_2(\lambda)$, we have $h(x)=f_1(x)f_2(x)$, provided that $x \smile f_2(\lambda)$ for each λ in Δ_2.

(P_4) Let $(\sigma(x),\Delta,f_k)$ be a sequence of triplets. Suppose that $f(\lambda)=\lim_k f_k(\lambda)$ exists in the norm topology of A, for each $\lambda\in\Delta$, uniformly with respect to λ in every compact subset of Δ. Then $f(x)=\lim f_k(x)$ in the norm topology of A.

The proofs of Properties (P_0), (P_3)–(A), and (P_4) are identical with the corresponding proofs in the case of scalar-valued functions (see [4, p. 169]). Properties (P_1) and (P_2) are immediate. Finally, Property (P_3)–(B) is a consequence of the following.

THEOREM 1. *Let $(\sigma(x),\Delta_i,f_i)$ $(i=1,2)$ be two triplets. Form the triplet $(\sigma(x),\Delta,f)$ with $\Delta=\Delta_1\cap\Delta_2$ and $f(\lambda)=f_1(\lambda)f_2(\lambda)$. Then $f(x)=f_1(x)f_2(x)$ if and only if the following holds for $\Gamma=\Gamma(\sigma(x);\Delta)$:*

$$\int_\Gamma [f_1(x)-f_1(\lambda)]R(\lambda;x)[f_2(\lambda)x-xf_2(\lambda)]R(\lambda;x)d\lambda=0.$$

Proof. For $x,\ y\in A$ and $\lambda,\ \mu\in\rho(x)$, we have the identity

$$(1)\quad (\mu-\lambda)R(\lambda;x)\,y\,R(\mu;x)=R(\lambda;x)y-yR(\mu;x)+R(\lambda;x)(yx-xy)R(\mu;x).$$

Indeed, if z denotes the right-hand side of (1), we have

$$(\lambda e-x)z(\mu e-x)=y(\mu e-x)-(\lambda e-x)y+yx-xy=(\mu-\lambda)y.$$

Now, given $\Gamma_2=\Gamma_2(\sigma(x),\Delta)$, we may choose $\Gamma_1=\Gamma_1(\sigma(x),\Delta)$ such that $\Omega_2\cup\Gamma_2\subset\Omega_1$. We have

$$f_1(x)f_2(x)=\left(\frac{1}{2\pi i}\right)^2\int_{\lambda\in\Gamma_1}\int_{\mu\in\Gamma_2} f_1(\lambda)\ \{R(\lambda;x)f_2(\mu)R(\mu;x)\}\,d\lambda\,d\mu.$$

Using (1) with $y=f_2(\mu)$ and noting that for $\lambda\in\Gamma_1$ and $\mu\in\Gamma_2$ we have $\mu-\lambda\neq 0$, we obtain

$$\begin{aligned} {}_1(x)f_2(x) &= \frac{1}{2\pi i}\int_{\Gamma_1} f_1(\lambda)R(\lambda;x)\cdot\frac{1}{2\pi i}\int_{\Gamma_2}\frac{f_2(\mu)}{\mu-\lambda}\,d\mu\,d\lambda \\ &+ \frac{1}{2\pi i}\int_{\Gamma_2}\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f_1(\lambda)}{\lambda-\mu}\,d\lambda\cdot f_2(\mu)R(\mu;x)d\mu \\ &+ \left(\frac{1}{2\pi i}\right)^2\int_{\lambda\in\Gamma_1}\int_{\mu\in\Gamma_2}\frac{f_1(\lambda)}{\mu-\lambda}R(\lambda;x)\ [f_2(\mu)x-xf_2(\mu)]R(\mu;x)\,d\lambda\,d\mu \\ &= J_1+J_2+J_3. \end{aligned}$$

Since $\Omega_2\cup\Gamma_2\subset\Omega_1$ we conclude that

$$\frac{1}{2\pi i}\int_{\Gamma_2}\frac{f_2(\mu)}{\mu-\lambda}\,d\mu=0$$

for $\lambda \in \Gamma_1$ so that $J_1 = 0$. We also have

$$\frac{1}{2\pi i} \int_{\Gamma_1} \frac{f_1(\lambda)}{\lambda - \mu} d\lambda = f_1(\mu)$$

for $\mu \in \Gamma_2$, so that

$$J_2 = \frac{1}{2\pi i} \int_{\Gamma_2} f_1(\mu) f_2(\mu) R(\mu; x) d\mu = f(x).$$

Next, using the first resolvent equation [4, p. 126], we obtain

$$J_3 = \frac{1}{2\pi i} \int_{\Gamma_2} [f_1(x) - f_1(\mu)] R(\mu; x) [f_2(\mu)x - x f_2(\mu)] R(\mu; x) d\mu,$$

and the theorem follows.

REMARKS.

(a) Two special cases in which the condition of Theorem 1 holds are: (1) $f_1(\lambda) = f_1(x)$, i.e., $f_1(\lambda)$ is a constant (element of A); (2) $x \cup f_2(\lambda)$.

(b) The first resolvent equation is the special case of identity (1) with $y = e$.

We shall return to the problem of the multiplicativity of the map $f(\lambda) \to f(x)$ in §4.

2. **Spectral mapping.** The usual formulation [4, p. 171] of the Spectral Mapping Theorem has no meaning for an A-valued function $f(\lambda)$. If $f(\lambda) = \phi(\lambda)e$, with $\phi(\lambda)$ scalar-valued, then $f(x) = \phi(x)$ (if $(\sigma(x), \Delta, f)$ is a triplet). By the Spectral Mapping Theorem, $\sigma(f(x)) = \sigma(\phi(x)) = \phi(\sigma(x)) = \{\phi(\lambda);\ \lambda \in \sigma(x)\}$. But $\sigma(f(\lambda)) = \{\phi(\lambda)\}$ (for λ fixed in Δ). Thus:

(*) $$\sigma(f(x)) = \bigcup_{\lambda \in \sigma(x)} \sigma(f(\lambda)).$$

Written in this form, the statement of the Spectral Mapping Theorem makes sense for A-valued functions $f(\lambda)$. The question of the validity of equation (*) will now be studied.

THEOREM 2. *Let the triplet* $(\sigma(x), \Delta f)$ *be such that* $f(\lambda) \cup x$ *for all* λ *in* Δ. *Then*:

(1) $\sigma(f(x)) \subset \bigcup_{\lambda \in \sigma(x)} \sigma(f(\lambda))$.

(2) *For* $\mu \notin \bigcup_{\lambda \in \sigma(x)} \sigma(f(\lambda))$, *we have*

$$R(\mu; f(x)) = \frac{1}{2\pi i} \int_{\Gamma_\mu} R(\mu; f(\lambda)) R(\lambda; x) d\lambda$$

where $\Gamma_\mu = \Gamma(\sigma(x), \Delta(g_\mu))$ *and* $g_\mu(\lambda) = R(\mu; f(\lambda))$.

Proof. Let $\lambda_0 \in \Delta$, and let $\mu \in \rho(f(\lambda_0))$. Being holomorphic in a neighborhood of λ_0, the function $f(\lambda)$ is strongly continuous there; therefore we may choose $\delta > 0$ such that the sphere $S(\lambda_0; \delta) = \{\lambda;\ |\lambda - \lambda_0| \leq \delta\}$ is contained in Δ and

$$\|f(\lambda)-f(\lambda_0)\| < \frac{1}{2}\|R(\mu;f(\lambda_0))\|^{-1}$$

for all $\lambda \in S(\lambda_0;\delta)$. Thus, for $\lambda \in S(\lambda_0;\delta)$ (see [4, p. 127]),

$$R(\mu;f(\lambda)) = R(\mu;f(\lambda_0))\left\{e + \sum_{n=1}^{\infty}([f(\lambda)-f(\lambda_0)]R(\mu;f(\lambda_0)))^n\right\}; \tag{3}$$

hence:

$$\|R(\mu;f(\lambda))\| < 2\|R(\mu;f(\lambda_0))\|, \tag{4}$$

and, by the second resolvent equation (see [4, p. 127]),

$$\|R(\mu;f(\lambda)) - R(\mu;f(\lambda_0))\| \leqq 2\|R(\mu;f(\lambda_0))\|^2\|f(\lambda)-f(\lambda_0)\|.$$

We conclude from the strong continuity of $f(\lambda)$ in $S(\lambda_0;\delta)$ that $R(\mu;f(\lambda))$ is a strongly continuous function of λ in $S(\lambda_0;\delta)$, for μ as above.

Using again the second resolvent equation, we obtain (for $\lambda \in S(\lambda_0,\delta), \lambda \neq \lambda_0$)

$$\left\|\frac{R(\mu;f(\lambda)) - R(\mu;f(\lambda_0))}{\lambda-\lambda_0} - R(\mu;f(\lambda_0))f'(\lambda_0)R(\mu;f(\lambda_0))\right\|$$

$$\leqq 2\|R(\mu;f(\lambda_0))\|^2\left\|\frac{f(\lambda)-f(\lambda_0)}{\lambda-\lambda_0} - f'(\lambda_0)\right\|$$

$$+ \|f'(\lambda_0)\|\cdot\|R(\mu;f(\lambda_0))\|\cdot\|R(\mu;f(\lambda)) - R(\mu;f(\lambda_0))\| \to 0, \quad \lambda \to \lambda_0,$$

by the strong differentiability of $f(\lambda)$ at λ_0 (first term) and the strong continuity of $R(\mu;f(\lambda))$ at λ_0 (second term). This proves that $R(\mu;f(\lambda))$ is a strongly differentiable function of λ at any point $\lambda_0 \in \Delta$ such that $\mu \in \rho(f(\lambda_0))$. But, by (3), if $\mu \in \rho(f(\lambda_0))$, then also $\mu \in \rho(f(\lambda))$ for all $\lambda \in S(\lambda_0;\delta)$; thus $R(\mu;f(\lambda))$ is a strongly differentiable function of λ in a neighborhood of each $\lambda_0 \in \Delta$ such that $\mu \in \rho(f(\lambda_0))$. In particular, for $\mu \in \bigcap_{\lambda \in \sigma(x)}\rho(f(\lambda)), R(\mu;f(\lambda))$ is a strongly differentiable function of λ in a neighborhood of each $\lambda \in \sigma(x)$, and so in an open neighborhood Δ_μ of $\sigma(x)$. Hence $(\sigma(x),\Delta_\mu,g_\mu)$ is a triplet, and by Definition 1, we have

$$g_\mu(x) = \frac{1}{2\pi i}\int_{\Gamma_\mu} R(\mu;f(\lambda))R(\lambda;x)d\lambda, \tag{5}$$

where $\Gamma_\mu = \Gamma(\sigma(x),\Delta(g_\mu))$.

Next, for μ fixed, let $h_\mu(\lambda) = \mu e - f(\lambda)$ $(\lambda \in \Delta)$. We have

$$h_\mu(x) = \mu e - f(x);$$

$$g_\mu(\lambda)h_\mu(\lambda) = e \text{ for } \lambda \in \Delta_\mu;$$

$$x \cup g_\mu(\lambda);\ x \cup h_\mu(\lambda) \text{ for each } \lambda \in \Delta_\mu.$$

Therefore, by (P_3) and (P_1) we obtain

$$g_\mu(x)h_\mu(x) = h_\mu(x)g_\mu(x) = e,$$

i.e., by (5),

$$R(\mu;f(x)) = \frac{1}{2\pi i}\int_{\Gamma_\mu} R(\mu;f(\lambda))R(\lambda;x)\,d\lambda,$$

for each $\mu \in \bigcap_{\lambda\in\sigma(x)}\rho(f(\lambda))$. This proves (2). Taking complements, we obtain (1).

THEOREM 3. *Let the triplet $(\sigma(x),\Delta,f)$ be such that $x, f(\lambda)$ and $f(\lambda^1)$ commute for each λ and λ^1 in Δ. Let A_0 be the second commutant of x in A, and let $\mathscr{M}$ be its maximal ideal space. Then*:

$$\sigma(f(x)) = \{f(x(m))(m);\, m\in\mathscr{M}\}.$$

Proof. A_0 is a commutative Banach algebra containing e, x, $f(\lambda)$ (for each $\lambda\in\Delta$) and $f(x)$. According to [4, p. 203], the spectrum of an element y in A_0 as an element of A_0 is the same at the spectrum of y as an element of A.

Now, by the continuity of the Gelfand representation, we have, for each m in $\mathscr{M}$,

$$f(x)(m) = \frac{1}{2\pi i}\int_{\Gamma(\sigma(x),\Delta)} \frac{f(\lambda)(m)}{\lambda - x(m)}\,d\lambda.$$

But $f(\lambda)(m)$ is a locally holomorphic function of λ in Δ, and $x(m)\in\sigma(x)$ (see [4, p. 137]), Thus, by the Cauchy Integral Formula,

$$f(x)(m) = f(x(m))(m).$$

Now, for any $y\in A_0$, $\sigma(y)$ is the range of the function $y(m)$ ($m\in\mathscr{M}$). Thus:

$$\sigma(f(x)) = \{f(x)(m);\, m\in\mathscr{M}\} = \{f(x(m))(m);\, m\in\mathscr{M}\}.$$

REMARKS. (1) Under the hypothesis of Theorem 3, we have

$$\bigcup_{\lambda\in\sigma(x)} \sigma(f(\lambda)) = \{f(x(m^1))(m);\, m\in\mathscr{M},\, m^1\in\mathscr{M}\}.$$

In general, this set is larger than the set $\sigma(f(x)) = \{f(x(m))(m);\ m\in\mathscr{M}\}$. (See Remark (3) below.)

(2) A special case in which the two sets are equal is when the range of $x(m)$ or of $f(\lambda)(m)$ (for each λ in Δ), as functions of m ($m\in\mathscr{M}$), consists of a single complex number. For example, this happens when $x = \alpha e$, or x is generalized nilpotent, or $f(\lambda) = \phi(\lambda)e$, or $f(\lambda)$ is generalized nilpotent for each $\lambda\in\Delta$. (α and $\phi(\lambda)$ are complex.)

(3) A simple example for which we have proper inclusion in Theorem 2 (1) is obtained as follows:

Let A be the Banach algebra of all 2×2 matrices over the complex field, with the norm $\| x \| = \sup_{1 \leqq i,j \leqq 2} | x_{ij} |$. Let

$$f(\lambda) = \begin{pmatrix} \lambda + 1 & 0 \\ 0 & \lambda \end{pmatrix} \text{ and } x = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then $\sigma(f(x)) = \{1,2\}$, while $\bigcup_{\lambda \in \sigma(x)} \sigma(f(\lambda)) = \{0,1,2,3\}$.

(4) Under the hypothesis of Theorem 2, we have

$$r(f(x)) \leqq \sup_{\lambda \in \sigma(x)} r(f(\lambda)) \leqq \sup_{\lambda \in \sigma(x)} \| f(\lambda) \|.$$

If $| \mu | > \sup_{\lambda \in \sigma(x)} r(f(\lambda))$ and $\Delta(f)$ is the complex plane, then

$$R(\mu; f(x)) = \frac{1}{2\pi i} \int R(\mu; f(\lambda)) R(\lambda; x) \, d\lambda, \, | \lambda | = K,$$

where $| \mu | > K > \sup_{\lambda \in \sigma(x)} r(f(\lambda))$.

3. **Composite A-valued functions.** In this section, we discuss a generalization of the Composite Function Theorem (see [4, p. 171]) to A-valued functions.

Let f and g be two A-valued functions (conditions on $\Delta(f)$ and $\Delta(g)$ will be stated later). The composite function $h = g(f)$ is defined by $h(\lambda) = g(f(\lambda))$, whenever the right-hand side of this equation is meaningful in the sense of Definition 1. More precisely, $h(\lambda)$ is only defined for arguments λ such that g is locally holomorphic in an open neighborhood of $\sigma(f(\lambda))$. Now, let $x \in A$ and consider the elements $f(x)$, $g(f(x))$, and $h(x) = [g(f)](x)$ in the sense of Definition 1. The problem considered in this section is: under what conditions is it true that $g(f(x)) = [g(f)](x)$? A special case has been already discussed in Theorem 2. Indeed, if $g_\mu(\lambda) = (\mu - \lambda)^{-1} e$ (for μ fixed) and $f(\lambda)$ is an A-valued function locally holomorphic in $\Delta \supset \sigma(x)$, then we have

$$g_\mu(f(\lambda)) = R(\mu; f(\lambda)),$$

$$[g_\mu(f)](x) = \frac{1}{2\pi i} \int_{\Gamma_\mu} R(\mu; f(\lambda)) R(\lambda; x) \, d\lambda,$$

and

$$g_\mu(f(x)) = R(\mu; f(x)),$$

for suitable μ and Γ_μ. Thus, Theorem 2 states that $g_\mu(f(x)) = [g_\mu(f)](x)$, provided that x commutes with $f(\lambda)$ for all λ in Δ and that $\mu \notin \bigcup_{\lambda \in \sigma(x)} \sigma(f(\lambda))$.

The commutativity condition ($x \smile f(\lambda)$ for all $\lambda \in \Delta$) is essential in the problem. Indeed, let $f(\lambda) = \lambda e + y$, and $g(\lambda) = \lambda^2 e$. Suppose that x does not commute with y (i.e., x does not commute with $f(\lambda)$). Then $g(f(x)) = (x + y)^2$, while $[g(f)](x) = x^2 + 2yx + y^2 \neq g(f(x))$.

Next, we note that $[g(f)](x)$ is meaningful in the sense of Definition 1 only if $h = g(f)$ is locally holomorphic in an open neighborhood of $\sigma(x)$. From the considerations above, it follows that we must assume (at least) that g is locally holomorphic in an open neighborhood Δ_λ of $\sigma(f(\lambda))$ for each λ in an open neighborhood Δ of $\sigma(x)$. Thus $\Delta(g) \supset \bigcup_{\lambda \in \Delta} \sigma(f(\lambda))$ (where Δ is some open neighborhood of $\sigma(x)$). We shall require a little more, namely that $\Delta(g)$ includes the *closure* of $\bigcup_{\lambda \in \Delta} \sigma(f(\lambda))$, for some fixed open neighborhood Δ of $\sigma(x)$. We have:

THEOREM 4. *For $x \in A$, let $(\sigma(x), \Delta_1, f)$ and (Φ, Δ_2, g) be two triplets such that*:

(1) *Φ is the closure of $\bigcup_{\lambda \in \Delta} \sigma(f(\lambda))$, where Δ is an open set such that $\sigma(x) \subset \Delta \subset \Delta_1$.*

(2) *x commutes with $f(\lambda)$ for all $\lambda \in \Delta_1$. Then*:

$$g(f(x)) = [g(f)](x)$$

(*meaning that both elements exist and are equal*).

Proof. By Theorem 2.

$$\sigma(f(x)) \subset \Phi \subset \Delta_2.$$

Hence, the triplet $(\sigma(f(x)), \Delta_2, g)$ is meaningful, and, by Definition 1, $g(f(x))$ is well defined:

$$(3) \qquad g(f(x)) = \frac{1}{2\pi i} \int_{\Gamma_2} g(\mu) R(\mu; f(x))\, d\mu, \text{ where } \Gamma_2 = \Gamma(\sigma(f(x)), \Delta_2).$$

Now, the relation $\sigma(f(x)) \subset \Phi \subset \Delta_2$ implies that if $\Gamma = \Gamma(\Phi, \Delta_2)$, then also $\Gamma = \Gamma(\sigma(f(x)), \Delta_2)$. Then, *we choose* $\Gamma_2 = \Gamma(\Phi, \Delta_2)$.

We saw in the proof of Theorem 2 that, for fixed $\mu_0 \in \Gamma_2, R(\mu_0; f(\lambda))$ is a locally holomorphic function of λ in an open set $\Delta_{\mu_0} \supset \sigma(x)$. But for $|\mu - \mu_0| < \| R(\mu_0; f(\lambda)) \|^{-1}$ we have

$$R(\mu; f(\lambda)) = R(\mu_0; f(\lambda)) \cdot \left\{ e + \sum_{n=1}^{\infty} (\mu_0 - \mu)^n R(\mu_0; f(\lambda))^n \right\}$$

(see [4, p. 123]). This shows that $R(\mu; f(\lambda))$ is a locally holomorphic function of λ on Δ_{μ_0} for all μ in some neighborhood of μ_0. From the compactness of Γ_2, it follows that we can select a finite number of points $\mu_1, \cdots, \mu_n$ on Γ_2, such that $R(\mu; f(\lambda))$ is a locally holomorphic function of λ on the open set

$$\Delta_0 = \bigcap_{i=1}^{n} \Delta_{\mu_i} \cap \Delta \text{ for all } \mu \text{ on } \Gamma_2.$$

Since $\Delta_{\mu_i} \supset \sigma(x)$ $(i = 1, \cdots, n)$, we have $\Delta_0 \supset \sigma(x)$.

Let $\Gamma_1 = \Gamma(\sigma(x), \Delta_0)$. Then, for each μ on Γ_2, we have (by Theorem 2)

$$R(\mu; f(x)) = \frac{1}{2\pi i} \int_{\Gamma_1} R(\mu; f(\lambda)) R(\lambda; x)\, d\lambda. \tag{4}$$

From (3) and (4) we obtain

$$g(f(x)) = \frac{1}{2\pi i} \int_{\Gamma_2} g(\mu) \cdot \frac{1}{2\pi i} \int_{\Gamma_1} R(\mu; f(\lambda)) R(\lambda; x)\, d\lambda\, d\mu$$

$$= \frac{1}{2\pi i} \int_{\Gamma_1} \left\{ \frac{1}{2\pi i} \int_{\Gamma_2} g(\mu) R(\mu; f(\lambda))\, d\mu \right\} R(\lambda; x)\, d\lambda.$$

Consider the expression in braces for *fixed* $\lambda \in \Gamma_1$. We have

$$\sigma(f(\lambda)) \subset \Phi \subset \Delta_2.$$

Thus $(\sigma(f(\lambda)), \Delta_2, g)$ is a triplet. Furthermore, since $\Gamma_2 = \Gamma(\Phi, \Delta_2)$, then also $\Gamma_2 = \Gamma(\sigma(f(\lambda)), \Delta_2)$. We conclude that the expression in braces is $g(f(\lambda))$ in the sense of Definition 1. Hence

$$g(f(x)) = \frac{1}{2\pi i} \int_{\Gamma_1} g(f(\lambda)) R(\lambda; x)\, d\lambda. \tag{5}$$

We saw above that $R(\mu; f(\lambda))$ is a locally holomorphic function of λ in Δ_0 for each $\mu \in \Gamma_2$. Since $g(f(\lambda)) = (1/2\pi i) \int_{\Gamma_2} g(\mu) R(\mu; f(\lambda)) d\mu$, we conclude that $g(f(\lambda))$ is locally holomorphic in $\Delta_0 \supset \sigma(x)$. Therefore $(\sigma(x), \Delta_0, g(f))$ is a triplet. Since $\Gamma_1 = \Gamma(\sigma(x), \Delta_0)$ we conclude that the right-hand side of (5) equals $[g(f)](x)$. Q.E.D.

4. **Convolutive properties of the resolvent.** In this section, we discuss some convolutive properties of the resolvent, which are direct corollaries of Theorem 2. The results of this section appear to be useful in perturbation problems, in the explicit solution of some functional equations, and, furtheremore, additional information on the multiplicativity of the map $f(\lambda) \to f(x)$ is obtained.

COROLLARY 1. *Let x and y be two commuting elements of A. Then for each $\mu \notin \sigma(x) + \sigma(y)$ we have*

$$R(\mu; x + y) = \frac{1}{2\pi i} \int_{\Gamma} R(\mu - \lambda; y) R(\lambda; x)\, d\lambda$$

where $\Gamma_\mu = \Gamma(\sigma(x), \mu - \rho(y))$.

Proof. Take $f(\lambda) = \lambda e + y$ in Theorem 2.

COROLLARY 2. *Let x and y be two commuting elements of A with disjoint spectra. Then the element $x - y$ is nonsingular, and we have*

$$(x-y)^{-1} = \frac{1}{2\pi i}\int_{\Gamma(\sigma(x),\rho(y))} R(\lambda; y)R(\lambda; x)\, d\lambda.$$

Proof. Since $\sigma(x)$ and $\sigma(y)$ are disjoint, we have

$$0 \notin \sigma(x) - \sigma(y).$$

Thus, we take $\mu = 0$ and $-y$ instead of y in Corollary 1. Q.E.D.

COROLLARY 3. *Let x and y be two commuting elements of A. Then for each $\mu \notin \sigma(x)\cdot\sigma(y)$, we have*

$$R(\mu; xy) = \frac{1}{2\pi i}\int_{\Gamma} R\left(\frac{\mu}{\lambda}; y\right) R(\lambda; x)\frac{d\lambda}{\lambda}$$

where $\Gamma_\mu = \Gamma(\sigma(x), \{\lambda; \mu \in \rho(\lambda y)\})$ is chosen such that $0 \notin \Gamma_\mu$.

Proof. Take $f(\lambda) = \lambda y$ in Theorem 2.

We note that the integrals in Corollaries 1 and 3 look like convolutions. Nevertheless, those are not "usual" convolutions, since the integration is not over a group; moreover, the variable μ is not necessarily on the path of integration Γ_μ, which finally, depends on μ. We now deduce from Corollaries 1 and 3 some convolutive properties of the resolvent in the usual sense. For $R(\mu; x+y)$, we consider convolutions in the L_2-sense over the group R_1 of the real numbers; for $R(\mu; xy)$, the convolution is over the group of the unit circle.

Let $L_2 \otimes A$ denote the Banach space of (the equivalence classes of) (3) A-valued strongly measurable functions $f(\xi)$, $\xi \in R_1$, for which $\int_{R_1} \|f(\xi)\|^2 d\xi < \infty$, with the norm $\|f\| = (\int_{R_1} \|f(\xi)\|^2 d\xi)^{1/2}$. The convolution $f * g$ of two elements f and g of $L_2 \otimes A$ is given by

$$(f*g)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{R_1} f(\xi-\eta)g(\eta)\,d\eta \quad (\xi \in R_1).$$

We consider the A-valued functions $R_x^+(\xi)$ and $R_x^-(\xi)$ of the real variable ξ defined by

$$R_x^+(\xi) = \frac{1}{\sqrt{2\pi}} R(i\xi; x); \quad R_x^-(\xi) = \frac{1}{\sqrt{2\pi i}} R(\xi; x),$$

where $x \in A$.

Let Re $\sigma(x)$ denote, as usual, the orthogonal projection of $\sigma(x)$ on the real axis. Then we write:

$$r^+(x) = \sup \operatorname{Re} \sigma(x); \quad r^-(x) = \inf \operatorname{Re} \sigma(x).$$

Similarly, let $s^+(x) = \sup \operatorname{Im} \sigma(x)$ and $s^-(x) = \inf \operatorname{Im} \sigma(x)$.

Finally, we write $x_\lambda = x - \lambda e$ for λ complex.

(3) Two functions are in the same equivalence class if they differ only on a set of Lebesgue measure 0.

THEOREM 5. *Let x and y be two commuting elements of A. Then*:

(1) $$R^+_{x_\alpha + y_\beta} = R^+_{x_\alpha} * R^+_{y_\beta} \text{ for } \alpha > r^+(x) \text{ and } \beta > r^+(y).$$

(2) $$R^-_{x_{i\alpha} + y_{i\beta}} = R^-_{x_{i\alpha}} * R^-_{y_{i\beta}} \text{ for } \alpha > s^+(x) \text{ and } \beta > s^+(y).$$

All the functions in these identities are in $L_2 \otimes A$ for α and β as indicated above.

Proof. Let $\varepsilon > 0$. Then, for all μ in the half-plane $\operatorname{Re} \mu > r^+(x) + r^+(y) + \varepsilon$, we have

(3) $$R(\mu; x + y) = \frac{1}{2\pi i} \int_{\operatorname{Re} \lambda = r^+(x) + \varepsilon} R(\mu - \lambda; y) R(\lambda; x) \, d\lambda.$$

In order to prove (3), we consider the contour Γ_K composed of the segment $|\operatorname{Im} \zeta| \leqq K$ on the line $\operatorname{Re} \zeta = r^+(x) + \varepsilon$ and of the left half C_K of the circle with this segment as diameter (Γ_K is positively directed).

We have $\Gamma_K = \Gamma(\sigma(x), \mu - \rho(y))$ if $K > 2r(x) + \varepsilon$, and therefore, by Corollary 1,

(4) $$R(\mu; x + y) = \frac{1}{2\pi i} \int_{\Gamma_K} R(\mu - \lambda; y) R(\lambda; x) \, d\lambda.$$

For K large enough, the norm of the integral along C_K is less than $4\pi K^{-1}$. Thus we obtain (3) by letting $K \to \infty$ in (4).

Now, if $\alpha > r^+(x)$, we may take $\varepsilon = \alpha - r^+(x) > 0$ in (3). Then for μ in $\operatorname{Re} \mu \geqq \alpha + \beta$ $(\beta > r^+(y))$, we have

$$R(\mu; x + y) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} R(\mu - \lambda; y) R(\lambda; x) \, d\lambda.$$

In particular, for μ on the line $\operatorname{Re} \mu = \alpha + \beta$, this identity is equivalent to (1).

We prove (2) in the same way.

We bring now Corollary 3 to a "proper" convolution form. For $x \in A$ such that $r(x) < 1$, we consider the A-valued function $R^0_x(\xi)$ of the real variable ξ defined by $R^0_x(\xi) = R(e^{2\pi i \xi}; x)$.

The convolution in this case is the convolution of the group R_1/I (of the real numbers modulo the integers), i.e.,

$$(f * g)(\xi) = \int_0^1 f(\xi - \eta) g(\eta) \, d\eta.$$

THEOREM 6. *Let x and y be two commuting elements of A. Then*

$$R^0_{x/\alpha \cdot y/\beta} = R^0_{x/\alpha} * R^0_{y/\beta} \text{ for } \alpha > r(x) \text{ and } \beta > r(y)$$

(*the convolution is for the group R_1/I*).

Proof. Let $\alpha > r(x)$, $\beta > r(y)$ and $\rho \geqq \alpha\beta$. Let $\mu = \rho e^{i\theta}$; then $\mu \notin \sigma(x) \cdot \sigma(y)$. Let $\Gamma = \{\lambda; |\lambda| = \alpha\}$; then $\Gamma = \Gamma(\sigma(x); \{\lambda; \mu \in \rho(\lambda y)\})$. Applying Corollary 3 with $\Gamma_\mu = \Gamma$, we obtain

$$R(\rho e^{i\Theta}; xy) = \frac{1}{2\pi}\int_0^{2\pi} R\left(\frac{\rho}{\alpha}e^{i(\Theta-\phi)}; y)R(\alpha e^{i\phi}; x\right) d\phi$$

$$(\alpha > r(x), \beta > r(y), \rho \geqq \alpha\beta).$$

The theorem follows by taking $\rho = \alpha\beta$. Q.E.D.

We study now some implications of the validity of the identity in Corollary 1. Ultimately, we shall see that this identity implies necessarily that x commutes with y. In Corollaries 4 and 5, we assume that Corollary 1 is valid rather than assuming that $x \cup y$. This is done because these corollaries are used as lemmas when we prove later the necessity of the condition $x \cap y$ for the validity of Corollary 1.

The two following results are perturbation formulas which generalize a theorem of J. Schwartz [5].

We first prove two simple lemmas.

Lemma 1. *For $\mu \in \rho(x)$, we have*

$$r(R(\mu; x)) = [\text{dist}\{\mu; \sigma(x)\}]^{-1} \quad (x \in A).$$

Proof. Let A_0 be the second commutant of x in A, and let $\mathcal{M}$ be its maximal ideal space. Then:

$$\begin{aligned} r(R(\mu; x)) &= \sup_{m \in \mathcal{M}} |R(\mu; x)(m)| = \sup_{m \in \mathcal{M}} |\mu - x(m)|^{-1} \\ &= \sup_{\zeta \in \sigma(x)} |\mu - \zeta|^{-1} \\ &= [\inf_{\zeta \in \sigma(x)} |\mu - \zeta|]^{-1} \\ &= [\text{dist}\{\mu; \sigma(x)\}]^{-1}. \quad \text{Q.E.D.} \end{aligned}$$

Notation. For $x, y \in A$, let

$$\rho(x, y) = \{\mu; \text{dist}\{\mu; \sigma(x)\} > r(y)\}.$$

We have $\rho(x, y) = \rho(x)$ if and only if y is generalized nilpotent $(r(y) = 0)$.

The complement of $\rho(x, y)$ will be denoted by $\sigma(x, y)$.

Lemma 2. $\sigma(x) + \sigma(y) \subset \sigma(x, y)$.

Proof. We have

$$(**) \qquad \{\mu; \sigma(x) \subset \mu - \rho(y)\} = [\sigma(x) + \sigma(y)]'$$

where the prime denotes complementation. Now, if $\mu \in \rho(x, y)$, then $\text{dist}\{\mu; \sigma(x)\} > r(y)$, and therefore $\sigma(x)$ is in the complement of $\mu - \sigma(y)$, i.e., $\sigma(x) \subset \mu - \rho(y)$. According to (**), the lemma is proved.

Remark. In particular, if $x \cup y$, we have $\sigma(x + y) \subset \sigma(x, y)$.

Corollary 4. *Let $x, y \in A$ be such that Corollary 1 is valid (e.g.(4), $x \cup y$). Then, for $\mu \in \rho(x, y)$, we have*

$$R(\mu; x + y) = \sum_{n=0}^{\infty} R(\mu; x)^{n+1} \cdot y^n \quad (y^0 = e).$$

The series converges in norm, uniformly with respect to μ in any compact subset of $\rho(x, y)$.

Proof. For $\mu \in \rho(x, y)$ let Γ_μ be the circle $|\lambda| = \delta$ with $r(y) < \delta < \text{dist}\,\{\mu;\sigma(x)\}$. Then, $\Gamma_\mu = \Gamma(\sigma(y); \mu - \rho(x))$. Since $\mu \notin \sigma(x) + \sigma(y)$ by Lemma 2, we obtain from Corollary 1

$$(1) \qquad R(\mu; x + y) = \frac{1}{2\pi i} \int_{|\lambda| = \delta} R(\mu - \lambda; x) R(\lambda; y)\, d\lambda.$$

On the contour of integration, we have (using Lemma 1)

$$|\lambda| < \text{dist}\,\{\mu; \sigma(x)\} = [r(R(\mu; x))]^{-1} = \left[\lim_{n \to \infty} \| R(\mu; x)^n \|^{1/n}\right]^{-1}.$$

Hence, by (4.7.2), p. 123 in [4],

$$(2) \qquad R(\mu - \lambda; x) = R(\mu; x) \cdot \sum_{n=0}^{\infty} \lambda^n R(\mu; x)^n;$$

the series converges in norm, uniformly in λ on $|\lambda| = \delta$. The corollary follows by substituting (2) in (1) and integrating term-by-term. The convergence assertion follows from the estimate

$$\limsup_{n \to \infty} \| R(\mu; x)^n y \|^{1/n} \leqq r(R(\mu; x)) r(y) = \frac{r(y)}{\text{dist}\,\{\mu; \sigma(x)\}} < 1.$$

Remark. If $x \cup y$ and y is generalized nilpotent, the expansion in Corollary 4 is valid for all μ in $\rho(x)$.

Corollary 5. *Let $x, y \in A$ be such that Corollary 1 is valid (e.g. (5), $x \cup y$). Let $f(\lambda)$ be an A-valued function, locally holomorphic in an open neighborhood Δ of $\sigma(x, y) \cup \sigma(x + y)$. Then*

$$(1) \qquad f(x + y) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} y^n \quad (y^0 = e)$$

(*the series converges in norm*).

Proof. Let $\Gamma = \Gamma(\sigma(x, y) \cup \sigma(x + y), \Delta)$. Since Γ is a compact subset of $\rho(x, y)$, we have by Corollary 4

(4) After Corollary 6 is proved, we may replace "e.g." by "i.e."

(5) See preceding footnote; accordingly, the set $\sigma(x, y) \cup \sigma(x + y)$ is equal to $\sigma(x, y)$ under the hypothesis of Corollary 5.

(2) $$R(\mu; x+y) = \sum_{n=0}^{\infty} R(\mu; x)^{n+1} y^n \quad (\text{for all } \mu \in \Gamma),$$

where the series converges in norm, *uniformly* with respect to μ on Γ.

We also have $\Gamma = \Gamma(\sigma(x+y), \Delta)$. Therefore, by Definition 1,

(3) $$f(x+y) = \frac{1}{2\pi i} \int_{\Gamma} f(\mu) R(\mu; x+y) d\mu.$$

Substituting (2) in (3) and integrating term-by-term, we obtain (1).

COROLLARY 6. *Let $x, y \in A$. Then*

$$R(\mu; x+y) = \frac{1}{2\pi i} \int_{\Gamma(\sigma(x), \mu - \rho(y))} R(\mu - \lambda; y) R(\lambda; x)\, d\lambda$$

for each $\mu \notin \sigma(x) + \sigma(y)$ if and only if x and y commute.

Proof. The sufficiency of the condition $x \cup y$ is given by Corollary 1. In order to prove the necessity, we apply Corollary 5 with $f(\lambda) = \lambda^2 e$ and use the symmetry $f(x+y) = f(y+x)$.

As a consequence of Corollary 6, we obtain the following

THEOREM 7. *Suppose that the map $f(\lambda) \to f(x)$ given by Definition 1 is multiplicative on the algebra of functions $f(\lambda)$ such that $\Delta(f) \supset \sigma(x)$. Then x is in the center of A.*

Proof. For $y \in A$ and $\mu \notin \sigma(x) + \sigma(y)$, let

$$f_\mu(\lambda) = R(\mu - \lambda; y).$$

By (**) in Lemma 2, we have $\Delta(f_\mu) = \mu - \rho(y) \supset \sigma(x)$. Thus, by Definition 1,

(1) $$f_\mu(x) = \frac{1}{2\pi i} \int_{\Gamma_1} R(\mu - \lambda; y) R(\lambda; x)\, d\lambda$$

where $\Gamma_1 = \Gamma(\sigma(x),\ \mu - \rho(y))$.

Let $g_\mu(\lambda) = (\mu - \lambda)e - y$; then $g_\mu(x) = \mu e - (x+y)$. On the other hand

(2) $$g_\mu(x) = \frac{1}{2\pi i} \int_{\Gamma_2} [(\mu - \lambda)e - y] R(\lambda; x)\, d\lambda$$

where $\Gamma_2 = \Gamma(\sigma(x), p)$ and p denotes the complex plane.

By the multiplicativity of the map $f(\lambda) \to f(x)$, it follows from (1) and (2) that $f_\mu(x) = g_\mu(x)^{-1} = R(\mu; x+y)$ for all $\mu \notin \sigma(x) + \sigma(y)$. We conclude from (1) and Corollary 6 that x commutes with y; since y is arbitrary, the theorem follows.

Given $x, y \in A$, we may write down the following convolution-like integrals for $\mu \notin \sigma(x) + \sigma(y)$:

$$J_1 = \frac{1}{2\pi i}\int_{\Gamma(\mu,x,y)} R(\mu-\lambda;y)R(\lambda;x)\,d\lambda;$$

$$J_2 = \frac{1}{2\pi i}\int_{\Gamma(\mu,x,y)} R(\lambda;x)R(\mu-\lambda;y)\,d\lambda$$

$$J_3 = \frac{1}{2\pi i}\int_{\Gamma(\mu,y,x)} R(\mu-\lambda;x)R(\lambda;y)\,d\lambda;$$

$$J_4 = \frac{1}{2\pi i}\int_{\Gamma(\mu,y,x)} R(\lambda;y)R(\mu-\lambda;x)\,d\lambda,$$

where $\Gamma(\mu,x,y,) = \Gamma(\sigma(x),\mu-\rho(y))$.

Obviously, J_k $(k = 1,\cdots,4)$ are well-defined elements of A. If $x \cup y$, all are equal to $R(\mu;x+y)$. Conversely, if $J_k = R(\mu;x+y)$ for some k and all $\mu\notin\sigma(x)+\sigma(y)$, then $x\cup y$ and all the J_k are equal (see Corollary 6).

Proposition. *For $x,y\in A$ and $\mu\notin\sigma(x)+\sigma(y)$, let*

$$J(\mu;x,y) = \frac{1}{2\pi i}\int_{\Gamma(\sigma(x),\mu-\rho(y))} R(\mu-\lambda;y)R(\lambda;x)\,d\lambda.$$

Then $J_1 = J_4 = J(\mu;x,y)$ and $J_2 = J_3 = J(\mu;y,x)$.

Furthermore, $J(\mu;x,y) = J(\mu;y,x)$ for all $\mu\notin\sigma(x)+\sigma(y)$ if and only if x and y commute. In this case, and only in this case, $J(\mu;x,y) = R(\mu;x+y)$.

Proof. Writing $\lambda = \mu-\lambda'$ in the integral J_1, we obtain

$$J_1 = \frac{1}{2\pi i}\int_{\mu-\Gamma(\sigma(x),\mu-\rho(y))} R(\lambda';y)R(\mu-\lambda';x)\,d\lambda' = \frac{1}{2\pi i}\int_{\Gamma_1} R(\lambda;y)R(\mu-\lambda;x)d\lambda$$

where Γ_1 is $\mu-\Gamma(\sigma(x),\ \mu-\rho(y))$ with *reversed orientation.* We have $\Gamma_1=\Gamma(\sigma(y),\ \mu-\rho(x))$; hence $J_1=J_4$. Interchanging x and y, we obtain $J_2 = J_3$.

If $x\cup y$, Corollary 1 gives $J(\mu;x,y) = J(\mu;y,x) = R(\mu;x+y)$ for each $\mu\notin\sigma(x)+\sigma(y)$. Conversely, if $J(\mu;x,y) = J(\mu;y,x)$ (still for all $\mu\notin\sigma(x)+\sigma(y)$), then, for R large enough, we obtain

$$\frac{1}{2\pi i}\int_{|\mu|=R} \mu^2 J(\mu;x,y)\,d\mu = x^2+2xy+y^2,$$

while

$$\frac{1}{2\pi i}\int_{|\mu|=R} \mu^2 J(\mu;y,x)\,d\mu = y^2+2yx+x\ ,$$

so that $xy = yx$. Q.E.D.

Remark. We may consider $J(\mu;x,y)$ as some kind of generalized resolvent of $x+y$ in the noncommutative case (for $\mu\notin\sigma(x)+\sigma(y)$). For example, for suitable contours Γ we have

$$\frac{1}{2\pi i}\int_{\Gamma} e^{\mu} R(\mu; x+y)\, d\mu = e^{x+y}$$

while

$$\frac{1}{2\pi i}\int_{\Gamma} e^{\mu} J(\mu; x, y)\, d\mu = e^{y}\cdot e^{x}$$

and

$$\frac{1}{2\pi i}\int_{\Gamma} e^{\mu} J(\mu; y, x)\, d\mu = e^{x}\cdot e^{y}.$$

We consider now some applications of Corollary 1 to spectral operators. We refer to [3] for definitions and properties.

COROLLARY 7. *Let S be a bounded spectral operator of scalar type on the Banach space X, and let $E(\cdot)$ be its resolution of the identity. Let $F(\lambda)$ be an operator-valued function, locally holomorphic in an open neighborhood Δ of $\sigma(S)$, and suppose that $S \cup F(\lambda)$ (for all $\lambda \in \Delta$). Then, for $\mu \notin \bigcup_{\lambda\in\sigma(S)}\sigma(F(\lambda))$, we have*

$$(*)\qquad R(\mu; F(S)) = \int R(\mu; F(\lambda))\, d\, E(\lambda).$$

Proof. If $\mu \notin \bigcup_{\lambda\in\sigma(S)}\sigma(F(\lambda))$, then, by Theorem 1,

$$R(\mu; F(S)) = \frac{1}{2\pi i}\int_{\Gamma_\mu} R(\mu; F(\lambda)) R(\lambda; S)\, d\lambda,$$

where $\Gamma_\mu = \Gamma(\sigma(S), \Delta(g_\mu))$ and $g_\mu(\lambda) = R(\mu; F(\lambda))$. Now

$$R(\lambda; S) = \int_{\sigma(S)} (\lambda - \zeta)^{-1}\, dE(\xi) \qquad (\text{for all } \lambda \in \rho(S)).$$

Thus, by Fubini's Theorem,

$$R(\mu; F(S)) = \frac{1}{2\pi i}\int_{\sigma(S)} \left\{\frac{1}{2\pi i}\int_{\Gamma_\mu} \frac{1}{\lambda-\zeta} R(\mu; F(\lambda))\, d\lambda\right\} dE(\zeta).$$

Since $\zeta \in \sigma(S)$, it follows from the Cauchy Integral Theorem that the expression in braces is $R(\mu; F(\zeta))$. Q.E.D.

REMARK. Let S and $E(\cdot)$ be as in Corollary 7, and let L be an arbitrary bounded linear operator (on X into X) commuting with S. Then

$$R(\mu; S+L) = \int R(\mu-\lambda; L)\, dE(\lambda) \text{ for } \mu \notin \sigma(S) + \sigma(L)$$

and

$$R(\mu; SL) = \int R\left(\frac{\mu}{\lambda}; L\right) \frac{dE(\lambda)}{\lambda}$$

for $\mu \notin \sigma(S)\cdot\sigma(L)$. The integration is on any Borel set including $\sigma(S)$. (Take $F(\lambda) = \lambda I + L$ and λL respectively in Corollary 7.)

COROLLARY 8. *Let T be a spectral operator; let $E(\cdot)$ and N be the resolution of the identity and the generalized nilpotent part of T (respectively). Then, for each $\mu \in \rho(T)$,*

$$R(\mu; T) = \int_{\sigma(T)} R(\mu - \lambda; N)\, dE(\lambda).$$

Proof. Apply the above remark to the canonical decomposition of spectral operators.

COROLLARY 9. *The Spectral Theorem for normal operators.*

We sketch a proof of the spectral theorem for normal operators which is based on the above remark.

If N is a normal operator, then $N = A + iB$ where A and B are commuting hermitian operators; therefore, for $\mu \notin \sigma(A) + i\,\sigma(B)$, we have

$$(1)\quad R(\mu; N) = \int_{\sigma(A)} R(\mu - \xi; iB)\, dE_A(\xi) = \int_{\sigma(A)}\int_{\sigma(B)} \frac{1}{\mu - (\xi + i\eta)} dE_B(\eta) dE_A(\xi).$$

Next, if $f(\mu)$ is locally holomorphic in a neighborhood Δ of $\sigma(A) + i\,\sigma(B)$, and $\Gamma = \Gamma(\sigma(A) + i\,\sigma(B), \Delta)$, then we obtain from (1)

$$f(N) = \int_{\sigma(A)}\int_{\sigma(B)} f(\xi + i\eta) dE_B(\eta) dE_A(\xi). \qquad \text{Q.E.D.}$$

This proof arrives deductively at the fact that the spectral measure of N is the product measure $E_A \times E_B$.

A similar proof can be based on the polar decomposition $N = RU$ ($R = (NN^*)^{1/2}$ is hermitian; U is unitary), starting with the second identity in the same remark.

The next corollary deals with the perturbation of an arbitrary bounded linear operator by a compact operator.

COROLLARY 10. *Let B be an arbitrary bounded linear operator on the Hilbert space H. Let $A \neq 0$ be a compact hermitian operator commuting with B. Let $\{\lambda_k\}_{k=1}^n (1 \leqq n \leqq \infty)$ be the sequence of eigenvalues of A (each λ_k repeated according to its multiplicity, and, if $n = \infty$, $\lambda_\infty = 0$). Let x_k be a corresponding orthonormal sequence of eigenvectors of A.*

Then: (1) *for $\mu \notin \bigcup_{k=1}^n [\lambda_k + \sigma(B)]$, and all $y \in H$,*

$$R(\mu; A + B)y = R(\mu; B)\left\{y + \sum_{k=1}^{n} \lambda_k (y, x_k) R(\mu - \lambda_k; B)x_k\right\};$$

(2) *for $\mu \notin \bigcup_{k=1}^n \lambda_k \cdot \sigma(B)$,*

$$R(\mu; AB)y = \frac{1}{\mu}\left\{y + \sum_{k=1}^{n} (y, x_k) R\left(\frac{\mu}{\lambda_k}; B\right)x_k\right\}.$$

Proof. We prove only (1) in the case $n = \infty$. For $\mu \notin \bigcup_{k=1}^{\infty}[\lambda_k + \sigma(B)]$, we have by Corollary 1 and Theorem 6.4.C. in [6]

$$R(\mu; A + B) = \frac{1}{2\pi i} \int_{\Gamma_\mu} R(\mu - \lambda; B) \frac{d\lambda}{\lambda} y \tag{3}$$
$$+ \sum_{k=1}^{\infty} (y, x_k) \left\{ \frac{1}{2\pi i} \int_{\Gamma_\mu} \frac{\lambda_k}{\lambda(\lambda - \lambda_k)} R(\mu - \lambda; B)\, d\lambda \right\} x_k.$$

Since $\Gamma_\mu = \Gamma(\sigma(A), \mu - \rho(B))$, the Cauchy Integral Formula gives

$$\frac{1}{2\pi i} \int_{\Gamma_u} R(\mu - \lambda; B) \frac{d\lambda}{\lambda} = R(\mu; B)$$

(since $0 \in \sigma(A) \subset$ interior Γ_μ) and

$$\frac{1}{2\pi i} \int_{\Gamma_\mu} \frac{\lambda_k}{\lambda(\lambda - \lambda_k)} R(\mu - \lambda; B)\, d\lambda = \lambda_k R(\mu; B) R(\mu - \lambda_k; B)$$

(because $\lambda_k \in \sigma(A) \subset$ interior Γ_μ; $k = 1, 2, \cdots$). Substituting in (3), the corollary follows.

We consider now an application of Corollary 1 to a certain family of integral equations for functions of two variables.

Let f and g be in $L_2(R_2)$, where R_2 denotes the plane. The *composition* $f * g$ is defined by:

$$(f * g)(x, y) = \int_{R_1} f(x, t)\, g(t, y)\, dt.$$

Now, given $a, b \in L_2(R_2)$, we consider the integral equation

$$\mu f - a * f - f * b = g. \tag{I}$$

Let us define the operators A and B on $L_2(R_2)$ by

$$Af = a * f, \qquad Bf = f * b.$$

A and B are commuting bounded linear operators on $L_2(R_2)$. Applying Corollary 1, we thus obtain:

COROLLARY 11. *The problem of solving equation* (I) *is reduced to that of solving ordinary Fredholm equations as follows. We solve the ordinary Fredholm equations* $(\lambda I - A)f = g$ *(with the variable* y *as a parameter) and* $(\mu I - B)f = h$ *(with* x *as a parameter). Thus we obtain* $R(\lambda; A)$ *and* $R(\mu; B)$ $(\lambda \notin \sigma(A), \mu \notin \sigma(B))$. *Then the explicit solution of equation* (I) *is given by*

$$f = \frac{1}{2\pi i} \int_{\Gamma(\sigma(B), \mu - \rho(A))} R(\mu - \lambda; A) R(\lambda; B) g\, d\lambda,$$

for $\mu \notin \sigma(A) + \sigma(B)$.

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